Numerical Stability and Numerical Dispersion of a Compact 2-D/FDTD Method Used for the Dispersion Analysis of Waveguides

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Abstract— **The stability condition is derived for the compact two-dimensional FDTD (2-D/FDTD) scheme which was recently proposed for the dispersion analysis of waveguiding structures. It is shown that the upper limit of the Courant number depends on the desirable propagation constant β and is always smaller than the one for the standard FDTD scheme in two dimensions. The dispersion equation for the numerical scheme is derived also and is used to examine the impact of grid size on the accuracy of the calculated eigenvalues (frequencies) for the dominant and higher order modes.**

I. Introduction

In a recent paper [1], a compact 2-D/FDTD method was proposed for the dispersion analysis of uniform waveguiding structures. The proposed method, which is similar to the method presented in [2], takes advantage of the fact that for propagating modes the field variation along the axis of the waveguide, z, is of the form $\exp(-j\beta z)$, where β is the propagation constant and $j = \sqrt{-1}$. Thus, in Maxwell's equations the z derivatives are replaced with $-j\beta$ and the numerical discretization is restricted only on the cross-section of the waveguide. Fig. 1 depicts the unit cell of the 2-D lattice proposed in [1] for the discretization of Maxwell's equations over the cross-section of the waveguide. For example, the discrete form of the x component of Maxwell's first curl equation

$$\frac{\partial H_x}{\partial t} = -\frac{1}{\mu}\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right) \quad (1)$$

becomes

$$H_x^{n+1/2}(p, q+1/2) = H_x^{n-1/2}(p, q+1/2) - \frac{\Delta t}{\mu(p, q+1/2)}\left[\frac{E_z^n(p, q+1) - E_z^n(p, q)}{\Delta y} + j\beta E_y^n(p, q+1/2)\right], \quad (2)$$

where the notation $F^n(p, q) = F(p\Delta x, q\Delta y, n\Delta t)$ (p, q, n integers) has been adopted for the representation of the approximated field components. In (2), Δt is the time step, Δx is the cell size along x, Δy is the cell size along y, and central differences both in space and time have been used for the numerical approximation of the space and time derivatives. The rest of the scalar equations obtained from the two Maxwell's curl equations are discretized in a similar manner.

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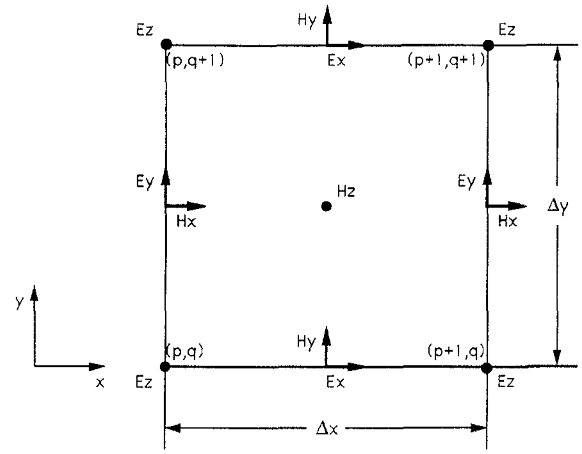

Fig. 1. The unit cell of the compact 2-D/FDTD lattice proposed in [1].

The solution of the discrete equations proceeds as follows. First, a desirable value for β is selected along with some initial value for the fields over the cross-section of the guide. Next, the equations are integrated in time and, subsequently, the time histories of the fields are Fourier-transformed to obtain the frequencies at which the various propagating modes in the guide exhibit the selected propagation constant β. These frequencies correspond to the peaks in the Fourier spectra.

In the following, the stability condition for this compact 2-D/FDTD scheme is derived. In addition, the numerical dispersion equation is obtained and used to comment on the impact of the cell size on the accuracy of the calculated eigenvalues (frequencies) for the various modes.

II. Stability Condition

The development of the stability condition follows the analysis presented in [3] for the standard FDTD scheme in a homogeneous, lossless medium with permittivity ϵ and permeability μ. In [3], it was shown that the numerical integration of Maxwell's equations on a rectangular FDTD lattice with cell size $\Delta x, \Delta y, \Delta z$ and time step Δt results in

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stable wave solutions of the form

$$\boldsymbol{E} = \boldsymbol{E}_0 \exp\left[j(\omega(n\Delta t) - \hat{k}_x(p\Delta x) - \hat{k}_y(q\Delta y) - \hat{k}_z(r\Delta z))\right] \quad (3a)$$

$$\boldsymbol{H} = \boldsymbol{H}_0 \exp\left[j(\omega(n\Delta t) - \hat{k}_x(p\Delta x) - \hat{k}_y(q\Delta y) - \hat{k}_z(r\Delta z))\right] \quad (3b)$$

provided that

$$\left(\frac{v\Delta t}{\Delta x}\right)^2 \sin^2\frac{\hat{k}_x\Delta x}{2} + \left(\frac{v\Delta t}{\Delta y}\right)^2 \sin^2\frac{\hat{k}_y\Delta y}{2} + \left(\frac{v\Delta t}{\Delta z}\right)^2 \sin^2\frac{\hat{k}_z\Delta z}{2} \leq 1, \quad (4)$$

where $v = (\mu\epsilon)^{-1/2}$, and $\hat{k}_x, \hat{k}_y, \hat{k}_z$ are the numerical wavenumbers along x, y, z, respectively. From (4) it is straightforward to obtain the familiar stability condition for the standard FDTD scheme

$$v\Delta t \leq \left[\frac{1}{(\Delta x)^2} + \frac{1}{(\Delta y)^2} + \frac{1}{(\Delta z)^2}\right]^{-\frac{1}{2}}. \quad (5)$$

For the compact 2-D/FDTD scheme, the derivative along z is calculated exactly and $\hat{k}_z = \beta$. Thus, taking the limit of the third term on the right-hand side of (4) as $\Delta z \to 0$ yields

$$\left(\frac{v\Delta t}{\Delta x}\right)^2 \sin^2\frac{\hat{k}_x\Delta x}{2} + \left(\frac{v\Delta t}{\Delta y}\right)^2 \sin^2\frac{\hat{k}_y\Delta y}{2} + \left(\frac{v\Delta t\beta}{2}\right)^2 \leq 1. \quad (6)$$

From (6), the stability condition for the compact 2-D/FDTD scheme is easily obtained

$$v\Delta t \leq \left[\frac{1}{(\Delta x)^2} + \frac{1}{(\Delta y)^2} + \left(\frac{\beta}{2}\right)^2\right]^{-\frac{1}{2}}. \quad (7)$$

It must be mentioned that this result was actually reported for the first time in [4] in the context of numerical simulations of transient electromagnetic wave propagation in inhomogeneous media.

III. Numerical Dispersion Equation

The numerical dispersion relation for the standard FDTD scheme is [5]

$$\sin^2\frac{\omega\Delta t}{2} = \left(\frac{v\Delta t}{\Delta x}\right)^2 \sin^2\frac{\hat{k}_x\Delta x}{2} + \left(\frac{v\Delta t}{\Delta y}\right)^2 \cdot \sin^2\frac{\hat{k}_y\Delta y}{2} + \left(\frac{v\Delta t}{\Delta z}\right)^2 \sin^2\frac{\hat{k}_z\Delta z}{2}. \quad (8)$$

In a manner similar to that used for deriving (7) from (4), the numerical dispersion relation for the compact 2-D/FDTD method is obtained from (8) by letting $\hat{k}_z = \beta$ and taking the limit of the third term on the right-hand side as $\Delta z \to 0$. This yields

$$\sin^2\frac{\omega\Delta t}{2} = \left(\frac{v\Delta t}{\Delta x}\right)^2 \sin^2\frac{\hat{k}_x\Delta x}{2} + \left(\frac{v\Delta t}{\Delta y}\right)^2 \cdot \sin^2\frac{\hat{k}_y\Delta y}{2} + \left(\frac{v\Delta t\beta}{2}\right)^2. \quad (9)$$

IV. Discussion

For simplicity, consider the case of a square lattice ($\Delta x = \Delta y = h$). Then (7) yields

$$\frac{v\Delta t}{h} \leq \left[2 + \left(\frac{\beta h}{2}\right)^2\right]^{-\frac{1}{2}}. \quad (10)$$

From (10), it is clear that, contrary to the statement in [1], the upper limit for the Courant number, $v\Delta t/h$, for the compact 2-D/FDTD is smaller than the familiar value of $1/\sqrt{2}$ which is obtained for the standard 2-D/FDTD scheme. It is only at cutoff ($\beta = 0$) where the Courant limit for the compact 2-D/FDTD method coincides with that for the standard 2-D/FDTD, as expected. However, if the cell size is chosen sufficiently small compared to the guide wavelength $\lambda_g = 2\pi/\beta$, the reduction in the Courant limit is negligible. Indeed, for $\beta h = 0.1$ the Courant limit is 0.70666525 compared to its value of 0.70710678 for the standard 2-D/FDTD.

The accuracy of the eigenfrequencies calculated using the compact 2-D/FDTD scheme can be analyzed on the basis of the numerical dispersion equation (9) as follows. For the sake of discussion, the case of a square air-filled metallic waveguide with perfectly conducting walls is considered. A square lattice of size h and N elements per side is used for the discretization of the cross-section of the guide. Thus, $Nh = a$, where a is the length of the side of the square cross-section. Since the tangential electric field is zero on the perfectly conducting walls, $\hat{k}_x(Nh) = l\pi$ and $\hat{k}_y(Nh) = m\pi$, where l, m are integers. Letting $s = c\Delta t/h$, where c is the speed of light in air, the numerical wavelength $\hat{\lambda}_{l,m}$ for the (l, m) mode with propagation constant β is obtained from (9)

$$\frac{\hat{\lambda}_{l,m}}{a} = \frac{\pi s}{N\sin^{-1}\left\{s\left[\sin^2\frac{l\pi}{2N} + \sin^2\frac{m\pi}{2N} + \left(\frac{\beta a}{2N}\right)^2\right]^{\frac{1}{2}}\right\}}. \quad (11)$$

From (11), it is straightforward to show that as $N \to \infty$ (and thus, $h \to 0$) $\hat{\lambda}_{l,m}$ approaches its exact value $\lambda_{l,m}$ given by

$$\frac{\lambda_{l,m}}{a} = 2\left[l^2 + m^2 + \left(\frac{\beta a}{\pi}\right)^2\right]^{-\frac{1}{2}}. \quad (12)$$

In order to demonstrate the impact of the cell size h on the accuracy of the calculated eigenfrequencies, (11) is used for the case of an air-filled square waveguide of side $a = 3$ cm. The propagation constant is taken to be $\beta = 209.4395\ \text{m}^{-1}$. Figs. 2 and 3 depict the variation with the number of cells, N, of the percentage relative error in $\hat{\lambda}_{l,m}$ defined as

$$\text{Relative Error} = \frac{\hat{\lambda}_{l,m} - \lambda_{l,m}}{\lambda_{l,m}} \times 100, \quad (13)$$

for $s = 0.6$ and $s = 0.25$, respectively, for various modes in the guide. From (11) and the fact that $\sin x \sim x$ is a very good approximation for $x = \pi/16$, one expects $\hat{\lambda}_{l,m}$ to be a very close approximation of $\lambda_{l,m}$ for $N > 8l$ and $N > 8m$, provided that $(\beta a/N) \ll 1$. Figs. 2 and 3 support this conjecture. In addition, while the accuracy of the numerical

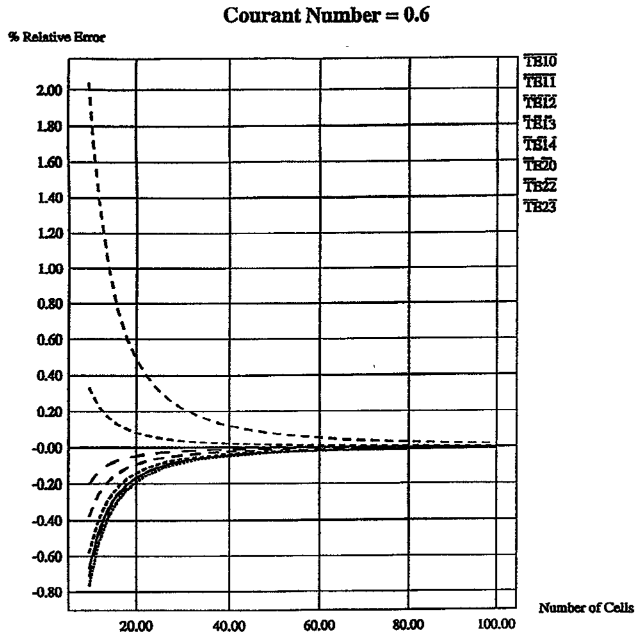

Fig. 2. Variation with the number of cells, N, of the percentage relative error in the numerical wavelength for various modes in an air-filled square waveguide of side $a = 3$ cm. The propagation constant is 209.4395 m^{-1} and the Courant number is 0.6.

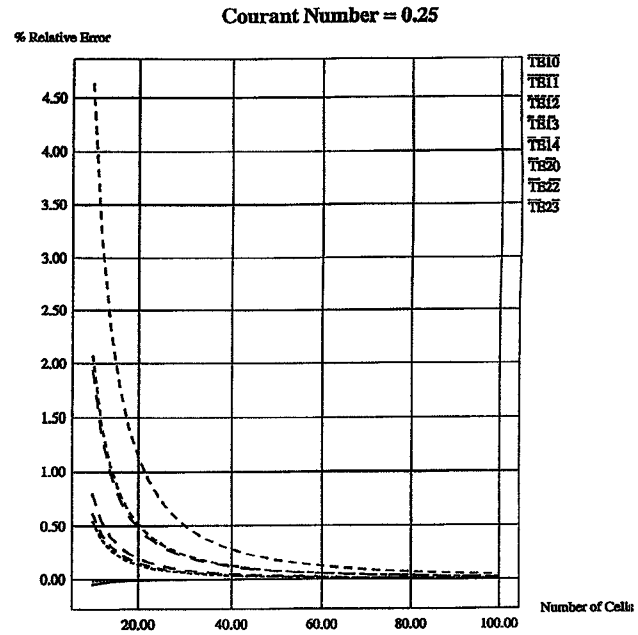

Fig. 3. Variation with the number of cells, N, of the percentage relative error in the numerical wavelength for various modes in an air-filled square waveguide of side $a = 3$ cm. The propagation constant is 209.4395 m^{-1} and the Courant number is 0.25.

wavelength for TE_{10} and TE_{11} is enhanced as s is decreased from 0.6 to 0.25, it is clear from Figs. 2 and 3 that the accuracy of the numerical wavelength for the higher-order modes deteriorates for low values of N.

V. Conclusion

In summary, this letter has considered the numerical stability and numerical dispersion of the compact 2-D/FDTD method proposed in [1] for the dispersion analysis of waveguiding structures. While the method removes the need for numerical discretization along the direction of propagation, the propagation constant β has still an impact on the proper choice of the grid size h in the discretization of the cross section of the guide. More specifically, if h is chosen such that $\beta h \ll 1$, the Courant limit for the compact 2-D/FDTD method is only slightly reduced from that for the standard 2-D/FDTD method. Furthermore, the numerical dispersion analysis for the simple case of an air-filled rectangular waveguide indicates that the choice $\beta h \ll 1$ is consistent with the requirements for calculating the eigenfrequencies with high accuracy.

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